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ON STABILITY OF STEADY MOTIONS OF A HEAVY SOLID BODY ON AN ABSOLUTELY SMOOTH HORIZONTAL PLANE*

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Stability of steady motions of a heavy solid on an absolutely smooth horizontal plane is investigated. Conditions of existence and stability of permanent rotations are obtained in the case of arbitrary distribution of mass and arbitrary surface of the body, and those of existence and stability of regular precessions of a dynamically symmetric body bounded by a surface of revolution. A definite analogy of the problem investigated here to that of stability of permanent rotations and regular precessions, respectively, of an arbitrary and a dynamically symmetric body with a fixed point, as well as the essential differences between them, are pointed out.

1. Consider a heavy solid body bounded by a smooth convex surface supported by an absolutely smooth horizontal plane. Position of the body is defined by the coordinates x and y of its center of mass in a fixed system of coordinates $Oxyz$, with the plane Oxy coincident with the horizontal supporting plane and the axis Oz directed vertically upward, and by the Euler angles θ, φ, ψ, between the principal central axes $G\xi, G\eta, G\zeta$ of the body ellipsoid of inertia and the axes of the fixed coordinate system. The Lagrange function of such system is then of the form

$$\begin{aligned}L = {}&{}^1/_2\,[A\cos^2\varphi + B\sin^2\varphi + m(\chi_1\cos\theta - \zeta\sin\theta)^2]\theta^{\cdot 2} + \\ &{}^1/_2\,(C + m\chi_2^2\sin^2\theta)\,\varphi^{\cdot 2} + {}^1/_2\,[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + \\ &C\cos^2\theta]\psi^{\cdot 2} + m(\chi_1\cos\theta - \zeta\sin\theta)\,\chi_2\sin\theta\,\theta^{\cdot}\varphi^{\cdot} + \\ &(A - B)\sin\theta\sin\varphi\cos\varphi\ \theta^{\cdot}\psi^{\cdot} + C\cos\theta\,\varphi^{\cdot}\psi^{\cdot} + {}^1/_2\, m\,(x^{\cdot 2} + \\ &y^{\cdot 2}) + mg\,(\chi_1\sin\theta + \zeta\cos\theta)\\ \chi_1 = {}&\xi\sin\varphi + \eta\cos\varphi,\ \ \chi_2 = \xi\cos\varphi - \eta\sin\varphi\end{aligned}$$

where m is the mass of the body, A, B, C are its principal central moments of inertia, and ξ, η, ζ are the coordinates of the point of contact of the body with the plane in the system of coordinates $G\xi\eta\zeta$. It can be shown that ξ, η, ζ are functions of vatiables θ and φ which are determined by the form of the two equations that define the body surface, and satisfy two relations of the form

$$(\xi'\sin\varphi + \eta'\cos\varphi)\sin\theta + \zeta'\cos\theta \equiv 0 \tag{1.1}$$

where the prime indicates differentiation with respect to θ or φ.

The coordinates x, y, ψ are obviously ignorable, and to them correspond the first integrals of the system

$$\frac{\partial L}{\partial x^{\cdot}} = p = \text{const},\quad \frac{\partial L}{\partial y^{\cdot}} = q = \text{const},\quad \frac{\partial L}{\partial \psi^{\cdot}} = Q = \text{const} \tag{1.2}$$

This enables us to disregard the ignorable variables and introduce the Routh function

$$\begin{aligned}R = {}&\frac{1}{2}\left[I_{22} - \frac{I_{23}^2}{I_{33}} + m(\chi_1\cos\theta - \zeta\sin\theta)^2\right]\theta^{\cdot 2} + \\ &\frac{1}{2}\left[I_{11} - \frac{I_{13}^2}{I_{33}} + m\chi_2^2\right]\sin^2\theta\varphi^{\cdot 2} + \left[-\left(I_{12} + \frac{I_{13}I_{23}}{I_{33}}\right) + \right.\\ &\left. m(\chi_1\cos\theta - \zeta\sin\theta)\chi_2\right]\sin\theta\,\theta^{\cdot}\varphi^{\cdot} + \frac{Q}{I_{33}}\,[-I_{23}\theta^{\cdot} + \\ &(I_{33}\cos\theta - I_{13}\sin\theta)\,\varphi^{\cdot}] + mg\,(\chi_1\sin\theta + \zeta\cos\theta) - \frac{1}{2}\frac{Q^2}{I_{33}} - \frac{1}{2m}(p^2 + q^2)\end{aligned}$$

where $I_{ij}\ (i, j = 1, 2, 3)$ are axial and $(i = j)$ centrifugal $(i \neq j)$ moments of inertia of the body about axes of the system of coordinates $Gx'y'z'$ whose origin is at the body center of mass, with the axis Gz' directed vertically upward, axis Gy' running on the line of nodes in the direction in which axis Gz' turns counterclockwise by angle θ up to congruence with axis $G\zeta$, and axis Gx' normal to plane $Gy'z'$, thus constituting a right-hand coordinate system

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$$\begin{aligned}
I_{11} &= (A\sin^2\varphi + B\cos^2\varphi)\cos^2\theta + C\sin^2\theta \\
I_{22} &= A\cos^2\varphi + B\sin^2\varphi \\
I_{33} &= (A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta \\
I_{12} &= (A - B)\sin\varphi\cos\varphi\cos\theta \\
I_{13} &= (A\sin^2\varphi + B\cos^2\varphi - C)\sin\theta\cos\theta \\
I_{23} &= -(A - B)\sin\varphi\cos\varphi\sin\theta
\end{aligned} \tag{1.3}$$

2. Since x, y, ψ are ignorable coordinates, the input system can perform steady motions of the form

$$\theta=\theta_0,\quad \theta^{\cdot}=0,\quad \varphi=\varphi_0,\quad \varphi^{\cdot}=0,\quad \psi^{\cdot}=\psi_0^{\cdot}\equiv\omega,\quad x^{\cdot}=x_0^{\cdot},\quad y^{\cdot}=y_0^{\cdot} \tag{2.1}$$

The body is then in contact with the horizontal plane at one and the same of its points, while rotating about the vertical line passing through its center of mass, with the latter moving at constant velocity along a straight line parallel to the horizontal plane. This means that the center of mass may be assumed, without loss of generality, stationary. The point of the body contact with the supporting plane describes on the latter a circle whose center is at the projection of the center of mass on that plane.

Constants $\theta_0, \varphi_0, \omega$ in (2.1) are determined by the system of three equations

$$\frac{\partial W}{\partial\theta}=0,\quad \frac{\partial W}{\partial\varphi}=0,\quad \frac{\partial L}{\partial\psi^{\cdot}}=Q$$
$$W=-mg(\chi_1\sin\theta+\zeta\cos\theta)+\frac{Q^2}{2I_{33}}$$

which with allowance for (1.1) and (1.3) assumes the form

$$(\xi\sin\varphi+\eta\cos\varphi)\cos\theta-\zeta\sin\theta=-\frac{Q^2 I_{13}}{mgI_{33}^2}$$
$$\xi\cos\varphi-\eta\sin\varphi=\frac{Q^2 I_{23}}{mgI_{33}},\quad I_{33}\omega=Q$$

Since constant Q is arbitrary in (1.2), ω can also be chosen arbitrarily, with constants θ_0 and φ_0 determined by the system of two equations of the form

$$\begin{aligned}
&(\xi\sin\varphi+\eta\cos\varphi)\cos\theta-\zeta\sin\theta=-\frac{I_{13}\omega^2}{mg} \\
&\xi\cos\varphi-\eta\sin\varphi=\frac{I_{23}\omega^2}{mg}
\end{aligned} \tag{2.2}$$

Eliminating from system (2.2) ω^2 we obtain the relation

$$(B-C)\xi\sin\theta\cos\varphi\cos\theta+(C-A)\eta\sin\theta\sin\varphi\cos\theta+(A-B)\zeta\sin^2\theta\sin\varphi\cos\varphi=0 \tag{2.3}$$

which in the case of steady motion must be satisfied by θ and φ or, which is the same, by the directional cosines $\gamma_1=\sin\theta\sin\varphi$, $\gamma_2=\sin\theta\cos\varphi$, $\gamma_3=\cos\theta$ of possible axes of permanent rotation of a heavy body on an absolutely smooth horizontal plane. Obviously Eq.(2.3) is, apart the notation, the same as the respective equation which must be satisfied by the directional cosines of possible axes of permanent rotations of a heavy solid with a fixed point /1/. But in the latter case ξ, η, ζ are the coordinates of the body center of mass in the system of principal axes of its ellipsoid of inertia about the fixed point, with A, B, C the corresponding principal moments of inertia. An essential difference between these two equations should be noted. In the case of a body with a fixed point ξ, η, ζ are constant and Eq.(2.3) defines in the space $\gamma_1, \gamma_2, \gamma_3$ a second-order cone, while in the considered here problem ξ, η, ζ are functions of θ and φ (i.e. functions $\gamma_1, \gamma_2, \gamma_3$) and Eq.(2.3) generally define an arbitrary surface.

Among the kinetically possible axes of permanent rotations whose cosines satisfy Eq.(2.3), not all but only those of them for which the inequality $\omega^2 \geqslant 0$ follows from (2.2). This implies the condition

$$(A-B)\sin\theta\sin\varphi\cos\varphi(\xi\cos\varphi-\eta\sin\varphi)\leqslant 0 \tag{2.4}$$

which apart the notation, is directly opposite to the condition that determines the admissible axes in the case of a solid with a fixed point /1/, i.e. with the given notation the axes admissible in the latter problem are inadmissible in our problem, and vice versa, which is obvious. In the case of permanent rotation of the body on the plane, its center of mass plays the part of fixed center, and the reaction force, equal and opposite to gravitation force, is applied to the body at its point of contact with the plane.

Remark. If the quantities ξ, η, ζ are compared with respective coordinates of the center of mass of a body with a fixed point taken with the opposite sign, then, apart such notation, condition (2.4) coincides with the corresponding condition in the case of a solid with a fixed point, and Eq.(2.3) remains unchanged.

3. Let us now consider the stability of permanent rotations of the input system. Using formulas (2.2) we reduce the Routh function to the form

$$R = {}^1/_2\,[au^{\cdot 2} + 2bu^{\cdot}v^{\cdot} + cv^{\cdot 2} + j\,(vu^{\cdot} - uv^{\cdot}) + du^2 + 2euv + fv^2] + \ldots \tag{3.1}$$

$$a = \left(I_{22} - \frac{I_{23}^2}{I_{33}} + \frac{I_{13}^2}{mg^2}\omega^4\right)_0$$

$$b = -\left(I_{12} + \frac{I_{13}I_{23}}{I_{33}} + \frac{I_{13}I_{23}}{mg^2}\omega^4\right)_0 \sin\theta_0$$

$$c = \left(I_{11} - \frac{I_{13}^2}{I_{33}} + \frac{I_{23}^2}{mg^2}\omega^4\right)_0 \sin^2\theta_0$$

$$j = \left(I_{11} + I_{22} - I_{33} - 2\,\frac{I_{13}^2 + I_{23}^2}{I_{33}}\right)_0 \omega\sin\theta_0$$

$$d = -\left[mg\,(r_1\cos^2\alpha + r_2\sin^2\alpha - h) + \omega^2\left(I_{33} - I_{11} + 4\,\frac{I_{13}^2}{I_{33}}\right)\right]_0$$

$$e = -\left[mg\,(r_2 - r_1)\sin\alpha\cos\alpha - \omega^2\left(I_{12} + 4\,\frac{I_{13}I_{23}}{I_{33}}\right)\right]_0 \sin\theta_0$$

$$f = -\left[mg\,(r_1\sin^2\alpha + r_2\cos^2\alpha - h) + \omega^2\left(I_{33} - I_{22} + 4\,\frac{I_{23}^2}{I_{33}}\right)\right]_0 \sin^2\theta_0$$

where u and v are the perturbations of variables θ and φ, $h = h(\theta, \varphi)$ is the height of the body center of mass above the supporting horizontal plane, $r_1 = r_1(\theta, \varphi)$ and $r_2 = r_2(\theta, \varphi)$ are the principal radii of curvature of the body surface at the point of contact with that plane, and $\alpha = \alpha(\theta, \varphi)$ is the angle between axis Gx' and the principal radius of curvature r_1 measured from axis Gx' toward axis Gy'; the subscript zero indicates that the respective function of variables θ and φ is determined for $\theta = \theta_0$, $\varphi = \varphi_0$, and the dotted line denotes terms of order not lower than the third of variables $u^{\cdot}$, $v^{\cdot}$ and u, v in the Routh function.

By the Routh theorem unperturbed motion is Liapunov stable with respect to $\theta, \theta^{\cdot}, \varphi, \varphi^{\cdot}, \psi^{\cdot}, x^{\cdot}, y^{\cdot}$, if the following conditions are satisfied:

$$d < 0 \text{ (or } f < 0) \tag{3.2}$$

$$df - e^2 > 0 \tag{3.3}$$

When inequalities (3.3) are strictly violated, the unperturbed motion is unstable according to the Kelvin–Chetaev theorem, while in the case of its fulfillment and strict violation of inequality (3.2) stability depends only on the sign of the expression

$$J = j^2 - (af + cd) + 2be - 2\,[(ac - b^2)(df - e^2)]^{1/2} \tag{3.4}$$

If $J < 0$, the characteristic equation of linearized equations of perturbed motion of the reduced system has a root in the right-hand half-plane, and the unperturbed motion is unstable, if however, $J > 0$ that characteristic equation has two pairs of pure imaginary roots. In that case the exact determination of the system stability requires further investigation.

Since θ_0 and φ_0 depend on ω^2 (see (2.2)), the stability of permanent rotations of a heavy solid on an absolutely smooth horizontal plane depends only on the distribution of mass and geometry of the body (*) surface and, also, on its angular velocity, while being independent of the direction of rotation (see (3.1)–(3.4)). If ω^2 is a eliminated from the stability condition with the use of (2.2), then for every specific form of the body surface it is possible to separate in the manifold (2.3), (2.4) of admissible axes of permanent rotations the domains of stable and unstable motions, as was done earlier in /1/ in the case of a solid with a fixed point. However this is not possible in the case of an arbitrary surface of the body, since the explicit form of functions $\xi(\theta, \varphi)$, $\eta(\theta, \varphi)$, $\zeta(\theta, \varphi)$ is not known.

Remarks. 1°. If the body rotates about one of the principal axes of its ellipsoid of inertia, the obtained here conditions of stability become the respective conditions obtained in /2/, if in the latter the rotor moment of momentum is assumed equal zero.

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*) Editor's note: in the original "of the field".

2°. When $\omega = 0$, Eqs.(2.2) determine the equilibrium position of a heavy solid on an absolutely smooth horizontal plane, and conditions (3.2) and (3.3) determine their stability. In that case $(\omega = 0)$, formulas (2.2) imply that in the equilibrium position the body center of mass lies on the vertical line which passes through the point of contact of the body and the support plane, and conditions (3.2) and (3.3) show that the equilibrium is stable (with respect to $\theta, \theta^{\cdot}, \varphi, \varphi^{\cdot}, \psi^{\cdot}, x^{\cdot}, y^{\cdot}$), if the body center of mass is below both principal centers of the body surface curvature at the point of its contact with the support plane, and unstable in the opposite case.

4. Example. Consider a heavy inhomogeneous sphere on an absolutely smooth horizontal plane. Let ρ be the sphere radius and a $-\lambda, -\mu, -\nu$ the coordinate of its geometric center in the system of coordinates $G\xi\eta\zeta$. Then

$$\xi = -\lambda - \rho\gamma_1, \quad \eta = -\mu - \rho\gamma_2, \quad \zeta = -\nu - \rho\gamma_3 \tag{4.1}$$

and Eq.(2.3) and condition (2.4) assume, respectively, the forms

$$(B - C)\lambda\gamma_2\gamma_3 + (C - A)\mu\gamma_1\gamma_3 + (A - B)\nu\gamma_1\gamma_2 = 0 \tag{4.2}$$

$$(A - B)\gamma_1\gamma_2(\lambda\gamma_2 - \mu\gamma_1) \geqslant 0 \tag{4.3}$$

which are exactly the same as the respective expressions in the case of the solid with a fixed point whose center of mass is defined by the coordinates λ, μ, ν relative to the principal axes of its ellipsoid of inertia about the fixed point. Equation (4.2) defines in that case also a second order cone. Thus within the indicated meaning of constants λ, μ, ν the manifolds of admissible axes of permanent rotations are the same in our problem and in that of motions of a solid with a fixed point.

Calculating the coefficients of the quadratic part of the Routh function for both cases and comparing them to each other, we obtain $a_1 = a_2 + \delta^2 = a$, $b_1 = b_2 - \delta\varepsilon = b$, $c_1 = c_2 + \varepsilon^2 = c$, $f_1 = f_2 = f$

$$d_1 = d_2 = -\left[-mgk + \omega^2\left(I_{33} - I_{11} + 4\frac{I_{13}^2}{I_{33}}\right)\right]_0$$

$$e_1 = e_2 = \omega^2\left(I_{12} + 4\frac{I_{13}I_{23}}{I_{33}}\right)_0 \sin\theta_0$$

$$f_1 = f_2 = -\left[-mgk + \omega^2\left(I_{33} - I_{22} + 4\frac{I_{23}^2}{I_{33}}\right)\right]_0 \sin^2\theta_0$$

$$\delta = \left(\frac{I_{13}\omega^2}{\sqrt{mg}}\right)_0, \quad \varepsilon = \left(\frac{I_{23}\omega^2}{\sqrt{mg}}\right)_0 \sin\theta_0$$

where a, b, c, f without subscripts are of form (3.1), subscripts 1 and 2 denote quantities in the investigated here problem and in that of the solid with a fixed point, respectively; in the first case $k = k(\theta, \varphi)$ is the z coordinate of the sphere geometric center, taken with the opposite sign, in the system of coordinates $Gx'y'z'$, and in the second it is the z coordinate of the body center of mass in the coordinate system with origin at the fixed point and similar to system $Gx'y'z'$.

This shows that, apart the indicated notation, the domain in which the sufficient conditions of stability of the heavy inhomogeneous sphere permanent rotations on an absolutely smooth horizontal plane, as determined by the Routh theorem, is the same as the respective domain in which sufficient conditions of stability of rotations of a heavy solid with a fixed point. A similar statement also holds for the domain where the sufficient conditions of instability determined by the Kelvin—Chetaev theorem are satisfied. As regards the domain of fulfillment of the necessary conditions of stability defined by the positive sign of J in (3.4) in the problem considered here, it is narrower than the respective domain in the case of the solid with a fixed point, since

$$J_1 - J_2 = -(f_2\delta^2 + 2e_2\delta\varepsilon + d_2\varepsilon^2) + 2(d_2f_2 - e_2^2)^{1/2}[(a_2c_2 - b_2^2)^{1/2} - (a_2c_2 - b_2^2 + c_2\delta^2 + 2b_2\delta\varepsilon + a_2\varepsilon^2)^{1/2}] < 0 \quad \forall \varepsilon,\ \delta,\ \varepsilon^2 + \delta^2 \neq 0$$

Note that when $A \neq B \neq C \neq A$, we have $\varepsilon = \delta = 0$ only if the rotation is about one of the principal axes of the ellipsoid of inertia of the body; all stability conditions are then the same in both problems (apart the indicated notation).

5. Consider now the case of a dynamically symmetric solid bounded by a surface of revolution. Taking the body axis of symmetry as the $G\zeta$, axis, we obtain for the system a Lagrange function of the form

$$L = {}^1/_2\,[A + m(\chi_1 \cos\theta - \zeta \sin\theta)^2]\,\theta^{\cdot 2} + {}^1/_2\,A \sin^2\theta\psi^{\cdot 2} +$$
$$ {}^1/_2\,C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta)^2 + {}^1/_2 m\,(x^{\cdot 2} + y^{\cdot 2}) + mg\,(\chi_1 \sin\theta + \zeta\cos\theta)$$

It can be shown that in such case χ_1 and ζ are functions of only variable θ, which satisfy the relation

$$\chi_1' \sin\theta + \zeta' \cos\theta = 0 \tag{5.1}$$

where the prime denotes differentiation with respect to θ, and $\chi_2 \equiv 0$.

Obviously x, y, φ, ψ are ignorable coordinates to which correspond the first integrals of the system

$$\frac{\partial L}{\partial x^{\cdot}} = p = \text{const},\quad \frac{\partial L}{\partial y^{\cdot}} = q = \text{const},\quad \frac{\partial L}{\partial \varphi^{\cdot}} = P = \text{const},\quad \frac{\partial L}{\partial \psi^{\cdot}} = Q = \text{const} \tag{5.2}$$

which enable us to disregard the ignorable variables and introduce the Routh function

$$R = {}^1/_2\,[A + m(\chi_1\cos\theta - \zeta\sin\theta)^2]\,\theta^{\cdot 2} + mg\,(\chi_1 \sin\theta + \zeta\cos\theta) - \frac{P^2}{2C} - \frac{1}{2m}(p^2 + q^2) - \frac{1}{2}\,\frac{(Q - P\cos\theta)^2}{A\sin^2\theta}$$

Moreover the input system can perform steady motions of the form

$$\theta = \theta_0,\quad \theta^{\cdot} = 0,\quad \varphi^{\cdot} = \varphi_0^{\cdot} \equiv \Omega,\quad \psi^{\cdot} = \psi_0^{\cdot} \equiv \omega,\quad x^{\cdot} = x_0^{\cdot},\quad y^{\cdot} = y_0^{\cdot} \tag{5.3}$$

The body then rotates about its axis of symmetry at constant angular velocity Ω and about the vertical line passing through its center of mass at constant velocity ω, and the center itself moves at constant velocity along a straight line parallel to the horizontal plane. As previously, we assume, without loss of generality, the center of mass to be stationary in steady motion. Then the point of contact of the body with the support plane describes in steady motion two circles: one on the body surface in a plane normal to its axis of symmetry and another on the support plane. The center of the first circle lies on the body axis of symmetry at a point at coordinate $\zeta(\theta_0)$, and the center of the other lies at the projection of the center of mass on the support plane.

The constants θ_0, Ω, ω in (5.3) are determined using the system of three equations

$$\frac{\partial W}{\partial \theta} = 0,\quad \frac{\partial L}{\partial \varphi^{\cdot}} = P,\quad \frac{\partial L}{\partial \psi^{\cdot}} = Q$$
$$W = -mg\,(\chi_1 \sin\theta + \zeta\cos\theta) + \frac{(Q - P\cos\theta)^2}{2A\sin^2\theta}$$

which with allowance for relations (5.1) assumes the form

$$-mg\,(\chi_1\cos\theta - \zeta\sin\theta) + \frac{(Q - P\cos\theta)(P - Q\cos\theta)}{A\sin^3\theta} = 0 \tag{5.4}$$
$$C(\Omega + \omega\cos\theta) = P,\quad A\sin^2\theta\,\omega + C(\Omega + \omega\cos\theta)\cos\theta = Q$$

Since constants P and Q in (5.2) are arbitrary, hence ω and Ω can also be arbitrarily chosen, with the constant θ_0 determined by the equation

$$-mg\,(\chi_1\cos\theta - \zeta\sin\theta) + [C\Omega + (C - A)\,\omega\cos\theta]\,\omega\sin\theta = 0 \tag{5.5}$$

The first equation of system (5.4) or Eq.(5.5) (obtained in /2/ in a somewhat different form) can be considered as the condition of existence of regular precession of a dynamically symmetric heavy solid bounded by a surface of revolution and supported by an absolutely smooth horizontal plane. In the case of the solid with a fixed point the corresponding condition of existence of regular precession (see, e.g., /3/) is obtained using (5.4) or (5.5) with $\chi_1 = 0$, $\zeta = -\nu$, where ν is the coordinate of the body center of mass on the axis of symmetry of its ellipsoid of inertia constructed for the fixed point.

6. In the considered here case (a single position coordinate in the system) the unperturbed motion is, obviously, stable when $(\partial^2 W/d\theta^2)_0 > 0$ and unstable, if the last inequality is strictly violated, which with (5.4) taken into account assumes the form

$$mg\,(r - l)_0 + A^{-1}\sin^{-4}\theta_0\,[(P - Q\cos\theta_0)^2 - 2(P - Q\cos\theta_0)\times(Q - P\cos\theta_0)\cos\theta_0 + (Q - P\cos\theta_0)^2] > 0 \tag{6.1}$$

where $r = r(\theta)$ is the radius of curvature of the meridian cross section of the body surface through the point of its contact with the support plane, $l = l(\theta)$ is the distance from that point to the point of intersection of the body axis of symmetry with the vertical line passing through the first point.

Note that the second term in condition (6.1) is always positive $(\theta \neq 0, \pi)$ and is exactly equal to the second derivative of the altered potential energy in the case of a solid with a fixed point (see, e.g., /3/), whose regular precession is always stable. However in this case it is not so. When $r_0 \geqslant l_0$, the regular precession of a solid on and absolutely smooth horizontal plane is also stable at any angular precession and spin velocities; but when $r_0 < l_0$, then as implied by (6.1) with allowance for (5.4), the inequality

$$[C\Omega + (C - 2A)\,\omega\cos\theta_0]^2 + A^2\omega^2\sin^2\theta_0 > -Amg\,(r_0 - l_0) \tag{6.2}$$

which imposes on ω and Ω constraints from below, must be satisfied.

Note that condition (6.2) is somewhat wider than the condition of stability of quasi-regular precession of a symmetric gyrostat on a horizontal plane, obtained in /2/ with the use of Liapunov's function; it is not only sufficient but, also, necessary.

Example. The regular precession of a dynamically symmetric heavy inhomogeneous sphere on an absolutely smooth horizontal plane is always stable, since then $r = l = \rho$, where ρ is the sphere radius.

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